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On the Interpolation by Discrete Splines with Equidistant Nodes

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In this paper we consider equidistant discrete splines $S(j)$, $j \in \mathbb{Z}$, which may grow as $O(|j|^s)$ as $|j| \to \infty$. Such splines are relevant for the purposes of digital signal processing. We give the definition of the discrete B-splines and describe their properties. Discrete splines are defined as linear combinations of shifts of the B-splines. We present a solution to the problem of discrete spline cardinal interpolation of the sequences of power growth and prove that the solution is unique within the class of discrete splines of a given order. © 2000 Academic Press

1. INTRODUCTION

The theory of cardinal interpolation is an essential topic in the spline studies, [8, 9]. The term cardinal interpolation means interpolation of a bi-infinite sequence by splines with equidistant nodes kh, $k \in \mathbb{Z}$. In the papers [8, 9, 12, 13] the authors studied cardinal interpolation by continuous polynomial splines. However, for the purposes of digital signal processing the discrete splines defined on the set $\mathbb{Z}$ of integers offer some advantages over the continuous ones. The discrete splines were studied in early seventies ([10]), but recently they reappeared as a subject of extensive investigations [1, 2 (Chapter 6), 4, 5, 6]. We mention also the related work [7] which deals with wavelets of discrete argument. Most part of the investigations were devoted to the theory of periodic discrete splines. In this paper we develop the theory of non-periodic discrete splines of power

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growth. The subject and methods involved are related to these of the work [8] where the slow growing continuous splines were studied.

The paper is organized as follows. Section 2 is devoted to the discrete B-splines. In Section 2.1 we give the definition of the B-spline B_p of an order p, establish its structure and outline its properties.

In Section 2.2 we introduce the characteristic cosine polynomials corresponding to discrete B-splines and prove its positivity. This result is basic for the solution of the cardinal interpolation problem.

In Section 3 we handle the problem of cardinal interpolation. First we define in Section 3.1 the discrete spline $S(j)$ as a linear combination of shifts of the B-spline. In Section 3.2 we present a solution to the problem and establish its uniqueness.

2. DISCRETE B-SPLINES

2.1. *Definition and Basic Properties of the B-Splines*

Splines we deal with are defined on the set of integers $\mathbb{Z}$. We start with the B-splines which are fundamental in any spline construction. Let p be a natural number. Throughout the paper we assume that n is an odd number.

The discrete B-spline of the first order is by definition the following sequence:

$$B_1(j) = \begin{cases} 1 & \text{if} \quad j \in 0 : n-1, \\ 0, & \text{otherwise}, \quad j \in \mathbb{Z}. \end{cases} \tag{1}$$

Here and further the script $l : m$ means the set of integers $\{l, l+1, ..., m\}$.

The higher order B-splines we define as the discrete convolutions by recurrence:

$$B_r = B_1 * B_{r-1}, \qquad r = 2, ..., p, \tag{2}$$

or, that is the same,

$$B_r(j) = \sum_{k=0}^{n-1} B_{r-1}(j-k), \qquad j \in \mathbb{Z}, \quad r = 2, ..., p. \tag{3}$$

It is readily seen that the B-spline of the second order is a piecewise polynomial of the first degree:

$$B_2(j) = \begin{cases} j+1, & \text{if} \quad j \in 0 : n-1 \\ 2n-1-j, & \text{if} \quad j \in n-1 : 2n-2, \\ 0, & \text{otherwise}, \quad j \in \mathbb{Z}. \end{cases} \tag{4}$$

In fact, any discrete B-spline is a piecewise polynomial. To prove this we use the z-transform [3].

DEFINITION 2.1. Let $f=\{f(k)\}_{k=-\infty}^{\infty}$ be a truncated sequence that is $f(k)=0$ for all $k<0$. The z-transform of f is the function of the complex variable z:

$$\zeta[f]=F(z)=\sum_{k=0}^{\infty} f(k)\, z^k, \qquad |z|<\rho, \tag{5}$$

where ρ is the radius of convergence of the series.

We mention two properties of the z-transform which are important for us:

- First one is concerned with the discrete convolution:

$$\zeta[f * g]=\zeta[f]\,\zeta[g]. \tag{6}$$

- The second is the shifting property:

$$z^l\zeta[f(\cdot)]=\zeta[f(\cdot-l)]. \tag{7}$$

The symbol $k_+^{(l)}$ will denote truncated factorial polynomial:

$$k_+^{(l)}=\begin{cases} k(k+1)\cdots(k+l-1) & \text{if} \quad k\in 0:\infty \\ 0, & k<0, \quad k\in\mathbb{Z}. \end{cases} \tag{8}$$

Let $k_+^{(0)}=1$ for $k>0$ and $k_+^{(0)}=0$ for $k\leqslant 0$. The z-transforms of the polynomials are:

$$\zeta[k_+^{(l)}]=\frac{l!z}{(1-z)^{(l+1)}}. \tag{9}$$

It is readily seen that

$$B_1(j)=(j+1)_+^{(0)}-(j+1-n)_+^{(0)}, \qquad \zeta[B_1]=\frac{1-z^n}{1-z}.$$

This relation implies that the z-transform of the B-spline is

$$\zeta[B_p]=\sum_{j=0}^{p(n-1)} B_p(j)\, z^j=(1+z+z^2+\cdots+z^{n-1})^p. \tag{10}$$

So, $B_p(j)$ is the coefficient at z^j in the polynomial $(1+z+z^2+\cdots+z^{n-1})^p$.

THEOREM 2.1. *The B-spline of the order p is the piecewise polynomial of the degree $p-1$:*

$$B_p(j)=\frac{1}{(p-1)!}\sum_{r=0}^{p}(-1)^r\binom{p}{r}(j+1-rn)_+^{(p-1)}$$
$$=\Delta_n^p\left(\frac{(j+1-pn)_+^{(p-1)}}{(p-1)!}\right). \tag{11}$$

Proof. From (2.10) we have:

$$\zeta[B_p]=\frac{(1-z^n)^p}{(1-z)^p}=\frac{\sum_{r=0}^{p}(-1)^r\binom{p}{r}z^{rn}}{(1-z)^p}$$
$$=\sum_{r=0}^{p}(-1)^r\binom{p}{r}\frac{z^{rn-1}}{(p-1)!}\zeta[j_+^{(p-1)}]. \tag{12}$$

Hence, invoking (7), we derive (11). ∎

The breakpoints $\{kn\}$, $k\in\mathbb{Z}$, are called the nodes of the B-spline. The following properties of the B-splines B_p hold:

1.

$$B_p(p(n-1)-j)=B_p(j)\qquad \text{for all integer } j; \tag{13}$$

2.

$$B_p(j)>0\qquad \text{if}\quad j\in 0:p(n-1) \tag{14}$$
$$B_p(j)=0\qquad \text{otherwise.} \tag{15}$$

3.

$$B_p(0)=B_p(p(n-1)=1; \tag{16}$$

4. The sequence $B_p(j)$ increases strictly monotonously as $0\leqslant j\leqslant p(n-1)/2$ and decays as $p(n-1)/2\leqslant j\leqslant p(n-1)$;

5.

$$\sum_{j\in\mathbb{Z}}B_p(j)=n^p. \tag{17}$$

The last assertion follows from (10) when $z=1$.

Remark. We emphasize that the B-splines assume only integer non-negative values and their supports are compact (Property 2). It is worth to note that the discrete B-spline $B_p(j)$ is not a trace of a continuous B-spline.

2.2. *Characteristic Cosine Polynomial*

Recall that n is an odd number: $n=2\nu+1$. Together with the B-spline $B_p(j)$ we introduce the central B-spline

$$M_p(j) := B_p(j+p\nu). \tag{18}$$

It is apparent that the central B-spline is an even sequence with the support at $-p\nu$: $p\nu$ and its maximum at zero. It is a piecewise polynomial of the degree $p-1$ with its nodes at $nk+p\nu$. We emphasize also that the convolution property holds:

$$M_r = M_1 * M_{r-1}, \qquad r=2, ..., p. \tag{19}$$

LEMMA 2.1. *For all integers k, q the following relation holds*:

$$\sum_{j=-\infty}^{\infty} M_p(j-kn)\, M_p(j-qn) = M_{2p}((k-q)\, n). \tag{20}$$

Proof. The property (19) implies that $M_p * M_p = M_{2p}$ which, in turn, leads to (20). ∎

Now we define a cosine polynomial which is fundamental for the sequel. Denote $b_p(k)=M_p(kn)$. Recall, that $b_p(-k)=b_p(k)$ and $b_p(k)$ is nonzero only if $|k| \leqslant \mu = [p\nu/n] = [p(n-1)/2n]$. Here $[\alpha]$ means the integer part of the number α.

DEFINITION 2.2. The cosine polynomial

$$T_p(x) = \sum_{k=-\mu}^{\mu} b_p(k)\, e^{ikx} = b_p(0) + 2 \sum_{k=1}^{\mu} b_p(k) \cos kx \tag{21}$$

we will call the characteristic cosine polynomial (CCP) of the B-spline M_p. It is related to the Euler–Frobenius polynomial ([9]).

It is apparent that $T_p(x)$ is an even 2π-periodic infinitely differentiable function. The basic property of the CCP is that it is strictly positive for all x. To establish that we first should prove the following assertion.

LEMMA 2.2. *Let m be an even positive number. Then for all $\lambda \in 1:m/2$ and natural p the function*

$$G_m(\lambda, p) := \sum_{s=0}^{n-1} \left(\frac{(-1)^s}{\sin \dfrac{\pi(sm+\lambda)}{mn}} \right)^p \tag{22}$$

is strictly positive and the following inequalities hold:

$$G_m(\lambda, p) \geqslant \begin{cases} 1, & \text{if} \quad p \text{ is odd} \\ \left(\sin \dfrac{\pi\lambda}{mn} \right)^{-p}, & \text{if} \quad p \text{ is even} \end{cases} \tag{23}$$

Proof. The estimate for the even exponents p is readily seen, since in this case all terms of the sum $G_m(\lambda, p)$ are positive and, therefore, the value of the sum exceeds its first term which is $(\sin(\pi\lambda/mn))^{-p}$. For odd p the situation is more complicated.

The function $q_\lambda(x) = \sin(\pi(xm+\lambda)/mn)$ has its only maximum on the interval $[0, n-1]$ at the point $x_0 = n/2 - \lambda/m$. On the intervals $[0, x_0]$ and $[x_0, n-1]$ the function is strictly monotonous. This fact implies that the minimal term of the positive sequence

$$h_\lambda(s) = \left(\sin \frac{\pi(sm+\lambda)}{mn} \right)^{-1}, \qquad s \in 0:n-1$$

is $h_\lambda(\nu)$, where $\nu = (n-1)/2$ and the subsequences $\{h_\lambda(s)\}_{s=0}^{\nu}$ and $\{h_\lambda(s)\}_{s=\nu+1}^{n-1}$ are strictly monotonous.

Let us return to the sum $G_m(\lambda, p)$. The cases when ν is odd or even require slightly different considerations.

1. In the case when ν is even we write the sum as follows:

$$\begin{aligned} G_m(\lambda, p) &= \sum_{s=0}^{n-1} ((-1)^s h_\lambda(s))^p \\ &= \sum_{s=0}^{\nu-1} ((-1)^s h_\lambda(s))^p + h_\lambda(\nu)^p + \sum_{s=\nu+1}^{n-1} ((-1)^s h_\lambda(s))^p. \end{aligned} \tag{24}$$

Due to monotonicity the sums in (24) are positive and we have

$$G_m(\lambda, p) > h_\lambda(\nu)^p \geqslant 1. \tag{25}$$

2. When ν is odd we write the sum as

$$G_m(\lambda, p) = \sum_{s=0}^{\nu} ((-1)^s h_\lambda(s))^p + h_\lambda(\nu+1)^p + \sum_{s=\nu+2}^{n-1} ((-1)^s h_\lambda(s))^p.$$

Hence we derive the inequality

$$G_m(\lambda, p) > h_\lambda(v+1)^p > h_\lambda(v)^p \geqslant 1. \quad \blacksquare \tag{26}$$

Now we proceed to establishing the basic property of the CCP.

THEOREM 2.2. *The cosine polynomial $T_p(x)$ is strictly positive for all x.*

Proof. Let us choose some even integer m subject to the inequality $m \geqslant 2\mu + 2$. Denote $\omega_m = e^{2\pi i/m}$. Then

$$T_p\left(\frac{2\pi l}{m}\right) = \sum_{k=-\mu}^{\mu} b_p(k)\, \omega_m^{-kl} = \sum_{k=-m/2}^{m/2-1} b_p(k)\, \omega_m^{-kl} = F_m(b_p)(l).$$

Here $F_m(b_p)$ denotes the m-point discrete Fourier transform (DFT) of the sequence b_p. We represent the function in an explicit form. To do that, we denote $N = mn$ and find the N-point DFT of the sequence $\{M_p(j)\}_{j=-N/2}^{N/2-1}$.

For the first order B-splines we have with $l \in -N/2 : N/2 - 1$:

$$u(l) := F_N(M_1)(l) = \sum_{j=-N/2}^{N/2-1} M_1(j)\, \omega_N^{-jl} = \sum_{j=-v}^{v} 1 \cdot \omega_N^{-jl}$$

$$= \begin{cases} 2v + 1 = n, & l = 0, \\ \dfrac{\sin \pi l/m}{\sin \pi l/N}, & l \neq 0. \end{cases}$$

Due to the convolution property (19),

$$F_N(M_p)(l) = [F_N(M_1)(l)]^p = u^p(l).$$

Let us extend periodically the sequence $u(l)$ with the period N. Then $u(sm) = 0$ when $s \in 1 : n-1$ and

$$M_p(j) = \frac{1}{N} \sum_{l=0}^{N-1} u^p(l)\, \omega_N^{lj}, \qquad j \in -N/2 : N/2 - 1. \tag{27}$$

Hence we have for $k \in -\mu : \mu$:

$$b_p(k) = M_p(kn) = \frac{1}{N} \sum_{l=0}^{N-1} u^p(l)\, \omega_m^{lk}.$$

Representing l as $l = sm + r$, $s \in 0 : n-1$, $r \in 0 : m-1$, we come to the relation:

$$b_p(k) = \frac{1}{m} \sum_{r=0}^{m-1} \left[\frac{1}{n} \sum_{s=0}^{n-1} u^p(sm + r) \right] \omega_m^{rk}. \tag{28}$$

For even integers p, Eq. (28) was established in [5]. Eq. (28) implies that

$$T_p\left(\frac{2\pi\lambda}{m}\right) = F_m(b_p)(\lambda) = \frac{1}{n}\sum_{s=0}^{n-1} u^p(sm+\lambda)$$

$$= \begin{cases} \frac{1}{n}(\sin \lambda\pi/m)^p\, G_m(\lambda, p), & \lambda \in 1:m-1 \\ n^{p-1}, & \lambda = 0. \end{cases} \tag{29}$$

The function $G_m(\lambda, p)$ was defined in (22).

Suffice it to evaluate $T_p(2\pi\lambda/m)$ when $\lambda \in 1:m/2$. On the interval $(0, \pi/2)$ the inequalities $(2/\pi)\, x < \sin x < x$ are true. They result in estimates

$$\left(\frac{2\lambda}{m}\right)^p < \left(\sin\frac{\lambda\pi}{m}\right)^p, \qquad \left(\frac{mn}{\lambda\pi}\right)^p < \left(\sin\frac{\lambda\pi}{mn}\right)^{-p}. \tag{30}$$

We should distinguish again the cases when p is even or odd.

1. In the case of even p the estimates (25) and (23) lead us straightforward to the following inequality

$$T_p\left(\frac{2\pi\lambda}{m}\right) \geqslant \frac{1}{n}\left(\frac{2n}{\pi}\right)^p > 0. \tag{31}$$

2. In the case of odd p for G only the estimate $G_m(\lambda, p) \geqslant 1$ is available. Then we have

$$T_p\left(\frac{2\pi\lambda}{m}\right) \geqslant \frac{1}{n}\left(\frac{2\lambda}{m}\right)^p. \tag{32}$$

Increasing m we come to the estimate

$$T_p(x) \geqslant \frac{1}{n}\left(\frac{2n}{\pi}\right)^p, \qquad x \in (-\infty, \infty)$$

when p is even. For odd values p Eq. (32) implies that $T_p(x) \geqslant (1/n)(x/\pi)^p$ $\forall x \in [0, \pi]$. But $T_p(0) = n^{p-1}$ and, due to the continuity of T, there exists some $d > 0$ such that $T_p(x) \geqslant n^{p-1}/2$ $\forall x \in [0, d]$. Hence we see that for odd p the inequality holds

$$T_p(x) \geqslant \max\left\{\frac{1}{2} n^{p-1}, \frac{1}{n}(d/\pi)^p\right\} \quad \forall x \in [0, \pi]. \tag{33}$$

Since $T_p(2\pi - x) = T_p(x)$, the inequality (33) is true for all real x. ▮

COROLLARY 2.1. *The function* $V(x)=1/T_p(x)$ *is even,* 2π*-periodic and infinitely differentiable. It could be expanded into the Fourier series*

$$V(x)=\sum_{k=-\infty}^{\infty} v(k)\, e^{ikx}, \tag{34}$$

and the coefficients

$$v(k)=\frac{1}{2\pi}\int_{-\pi}^{\pi} V(x)\, e^{-ikx}\, dx=\frac{1}{\pi}\int_{0}^{\pi} V(x)\cos kx\, dx \tag{35}$$

are decaying faster than any power of $1/k$ *as* $k\to\infty$. *Namely, for any* $\beta>0$ *there exists a constant* $C(\beta)$ *such that*

$$|v(k)|\leqslant\frac{C(\beta)}{(1+|k|)^{\beta}}, \qquad k\in\mathbb{Z}. \tag{36}$$

Remark. Note that Eq. (29) implies the identity:

$$\sum_{l=-\infty}^{\infty} M_p(ln)=n^{p-1}. \tag{37}$$

3. DISCRETE SPLINES AND CARDINAL INTERPOLATION

3.1. *Definition of the Discrete Spline and Some Preliminaries*

DEFINITION 3.1. Any linear combination of the shifts of the central discrete B-spline $M_p(j)$:

$$S_p(j)=\sum_{l=-\infty}^{\infty} c(l)\, M_p(j-ln) \quad (j\in\mathbb{Z}) \tag{38}$$

we will call the discrete spline of the order p.

The B-spline is compactly supported. Hence, once j is fixed, the series in (38) comprises actually only a few non-zero entries. To be specific, if $j\in kn:(k+1)\,n-1$ then

$$S_p(j)=\sum_{l=k-\mu}^{k+\mu+1} c(l)\, M_p(j-ln), \qquad \mu=\left[\frac{pv}{n}\right]=\left[\frac{p(n-1)}{n}\right]. \tag{39}$$

Here $[\alpha]$ means the integer part of the number α. Therefore the series in (38) converges with any coefficients $c(l)$. Moreover, due to Eq. (37), if $j \in kn:(k+1)\,n-1$ then

$$|S_p(j)| \leqslant n^{p-1} \max\{|c(l)|\}, \qquad l = k-\mu : k+\mu+1. \tag{40}$$

Note that S_p coincides with a polynomial of the degree $p-1$ on the set $kn - pv:(k+1)\,n - pv$. The points $\{kn - pv\}$, $k \in \mathbb{Z}$, are called the nodes of the spline S_p. We will handle the interpolation problem within somewhat restricted class of discrete splines. Before proceeding with it we state some definitions and auxiliary facts.

Definition 3.2. We denote by $\mathbf{G}^s$ the space of sequences $\vec{a} = \{a(k)\}_{-\infty}^{\infty}$ which satisfy the requirement $|a(k)| \leqslant M(1+|k|^s)$ $\forall k \in \mathbb{Z}$ with a fixed integer s and a positive constant M. The space $\mathbf{G} := \bigcup_{s=-\infty}^{\infty} \mathbf{G}^s$ is said to be the space of sequences of power growth.

Definition 3.3. We denote by $\mathbf{V}_p^s$ the space of discrete splines S_p such that the sequences $\{c(k)\}_{-\infty}^{\infty}$ in the representation (38) belong to $\mathbf{G}^s$ and the space $\mathbf{V}_p$ we define as $\mathbf{V}_p = \bigcup_{s=-\infty}^{\infty} \mathbf{V}_p^s$.

Remark. We stress that any spline $S(j) \in \mathbf{V}_p^s$ belongs to the space $\mathbf{G}^s$ with respect to $j \in \mathbb{Z}$. This follows straightforward from (40).

Some remarks on periodic distributions. Let $\vec{a} = \{a(k)\}_{-\infty}^{\infty} \in \mathbf{G}$. Denote

$$\mathscr{F}(\vec{a}, x) = \sum_k e^{ikx} a(k). \tag{41}$$

This series is a 2π-periodic distribution [11, p. 331]. The numbers

$$a(k) = \frac{1}{2\pi} \langle \mathscr{F}(\vec{a}, x), e^{-ikx} \rangle$$

are called the Fourier coefficients of the distribution.

Definition 3.4. We denote by $\mathbf{D}^s$ the space of 2π-periodic distributions given by (41) with $\vec{a} \in \mathbf{G}^s$, and $\mathbf{D} := \bigcup_{s=-\infty}^{\infty} \mathbf{D}^s$. The space of 2π-periodic complex-valued infinitely differentiable functions we denote by $\mathbf{C}^\infty$.

Under the discrete convolution of two sequences $\vec{q}$ and $\vec{r}$ we mean the sum:

$$\vec{q} * \vec{r} = \{s(k)\} = \left\{\sum_{l} q(k-l)\, r(l)\right\}$$

The following assertion is readily verified.

PROPOSITION 3.1. *The discrete convolution with a sequence from* $\mathbf{G}^{-\infty} = \bigcap_{s=-\infty}^{\infty} G^s$ *maps the space* $\mathbf{G}^s$ *into itself.*

The proposition implies that, provided

$$\vec{q} \in \mathbf{G}^{-\infty}, \qquad \vec{r} \in \mathbf{G}^s, \qquad \text{and} \qquad \vec{s} = \vec{q} * \vec{r},$$

the series

$$\sigma(x) := \sum_{k} e^{ikx} s(k) = \mathscr{F}(\vec{s}, x)$$

is the distribution from the space $\mathbf{D}^s$ as well as $\mathscr{F}(\vec{r}, x)$.
This fact justifies the following

DEFINITION 3.5. The product of a distribution $\rho = \mathscr{F}(\vec{r}, \cdot)$ from $\mathbf{D}^s$ with a function $Q = \mathscr{F}(\vec{q}, \cdot)$ from $\mathbf{C}^\infty$ will be understood as follows:

$$Q(x)\, \mathscr{F}(\vec{r}, x) = \mathscr{F}(\vec{r} * \vec{q}, x) \in \mathbf{D}^s. \tag{42}$$

It corresponds with the conventional definition of the multiplication of a distribution by a function.

3.2. *Cardinal Interpolation Problem*

Let us formulate the problem.

Cardinal discrete spline interpolation problem (*CDSIP*) *of order* p. Given a sequence $\vec{z} = \{z(k)\}$ of power growth, find a discrete spline of order p $S_p \in \mathbf{V}_p$ subject to the equations:

$$S_p(kn) = z(k), \qquad k \in \mathbb{Z}. \tag{43}$$

To obtain the solution of the CDSIP, we will follow generally the classical scheme by Schoenberg [8, 9].

Fundamental splines. Let us define the spline of the order p:

$$L_p(j) := \sum_{l=-\infty}^{\infty} v(l)\, M_p(j - ln), \tag{44}$$

where $v(l)$ are the Fourier coefficients of the function $V = 1/T$, (see (35)).

PROPOSITION 3.2. *The spline L_p offers the following properties*:

1.

$$L_p(kn) = \begin{cases} 1 & \textit{if} \quad k = 0, \\ 0 & \textit{otherwise}. \end{cases} \tag{45}$$

2. *The B-spline M_p can be represented uniquely through the spline L_p*:

$$M_p(j) := \sum_{l=-\mu}^{\mu} b_p(l)\, L_p(j - ln). \tag{46}$$

3. *For any $\beta > 0$ there exists a constant $D(\beta)$ such that the inequality*

$$|L_p(j - kn)| \leqslant \frac{D(\beta)}{(1 + |k|)^{\beta}}, \qquad k \in \mathbb{Z} \tag{47}$$

holds uniformly by $j \in \mathbb{Z}$.

Proof. 1. Let us write the spline L_p at the points kn:

$$L_p(kn) = \sum_{l=-\infty}^{\infty} v(l)\, M_p((k-l)\, n) = \sum_{l=-\mu}^{\mu} v(k-l)\, b_p(l).$$

Invoking Eq. (2.35) we get:

$$\begin{aligned} L_p(kn) &= \frac{1}{2\pi} \sum_{l=-\mu}^{\mu} b_p(l) \int_{-\pi}^{\pi} V(x)\, e^{-i(k-l)x}\, dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} dx\, V(x)\, e^{-ikx} \sum_{l=-\mu}^{\mu} b_p(l)\, e^{ilx} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} dx\, V(x)\, T(x)\, e^{-ikx} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} dx\, e^{-ikx} = \begin{cases} 1 & \text{if} \quad k = 0, \\ 0 & \text{otherwise}. \end{cases} \end{aligned}$$

2. Using (44) we may write

$$\sum_{l=-\mu}^{\mu} b_p(l)\, L_p(j-ln) = \sum_{l=-\mu}^{\mu} b_p(l) \sum_{k=-\infty}^{\infty} v(k)\, M_p(j-(k+l)\,n)$$
$$= \sum_{r=-\infty}^{\infty} M_p(j-rn) \sum_{l=-\mu}^{\mu} b_p(l)\, v(r-l)$$
$$= \sum_{r=-\infty}^{\infty} M_p(j-rn)\, L_p(rn) = M_p(j).$$

To verify the uniqueness of the representation (46), suppose that there exists another representation

$$M_p(j) := \sum_{l=-\infty}^{\infty} q(l)\, L_p(j-ln).$$

But Eq. (45) implies that

$$q(l) = M_p(ln) = b_p(l).$$

3. The inequality (47) is an immediate consequence of the estimate (36). ▮

The spline L_p is called the fundamental spline. Now we are in a position to establish the main result of the paper.

THEOREM 3.1. *The CDSIP of order p has unique solution with any set of data* $\{z(k)\}$ *of power growth. The solution is given by the formulas*

$$S_p^i(j) = \sum_{-\infty}^{\infty} z(k)\, L_p(j-kn)$$
$$= \sum_{-\infty}^{\infty} c(k)\, M_p(j-kn), \tag{48}$$
$$c(k) = \sum_{-\infty}^{\infty} v(l)\, z(k-l). \tag{49}$$

Moreover, if the sequence $\vec{z} = \{z(k)\}$ *belongs to* $\mathbf{G}^s$ *then the discrete spline* S_p^i *belongs to the space* $\mathbf{V}^s$.

Proof. Let the data sequence $\vec{z}=\{z(k)\}$ be from $\mathbf{G}^s$. Then, due to the estimate (47), the series

$$J(j):=\sum_{-\infty}^{\infty} z(k)\, L_p(j-kn)$$

converges absolutely and locally uniformly with respect to j. The property (45) implies that

$$J(kn)=z(k).$$

Substituting (44) we may write

$$\begin{aligned} J(j) &= \sum_{-\infty}^{\infty} z(k) \sum_{l=-\infty}^{\infty} v(l)\, M_p(j-(l+k)\, n) \\ &= \sum_{-\infty}^{\infty} c(k)\, M_p(j-kn), \\ c(k) &= \sum_{-\infty}^{\infty} v(l)\, z(k-l). \end{aligned}$$

Proposition 3.1 guarantees that the sequence $\{c(k)\}$ belongs to $\mathbf{G}^s$, whence we conclude that J is a discrete spline of the order p from the space $\mathbf{V}_p^s$ which provides a solution to the CDSIP. We redenote

$$S_p^i(j):=J(j).$$

It remains to prove the unicity of the solution S_p^i within the class $\mathbf{V}_p$. Suppose, that a discrete spline

$$R(j)=\sum_{l=-\infty}^{\infty} d(l)\, M_p(j-ln)\in \mathbf{V}_p$$

interpolates the zero sequence:

$$R(kn)=0, \qquad k\in\mathbb{Z}. \tag{50}$$

Using (46) we rewrite the spline R:

$$R(j)=\sum_{l=-\infty}^{\infty} f(l)\, L_p(j-ln), \qquad f(l)=\sum_{k=-\mu}^{\mu} b_p(k)\, d(l-k).$$

The relation (50) is equivalent to the following one:

$$f(k)=0, \qquad k\in\mathbb{Z}. \tag{51}$$

We should prove that Eq. (51) implies that

$$d(k)=0, \qquad k \in \mathbb{Z}. \tag{52}$$

Actually, the array $\vec{f}=\{f(l)\}_{-\infty}^{\infty}$ is the convolution:

$$\vec{f}=\vec{d} * \vec{b}_p$$

where $\vec{d}=\{d(l)\}_{-\infty}^{\infty}$, $\vec{b}_p=\{b_p(l)\}_{-\infty}^{\infty}$. Note, that $\{b_p(l)\}_{-\infty}^{\infty}$ are the Fourier coefficients of the cosine polynomial T_p. Denote by $P(x)=\mathscr{F}(\vec{d}, x) =\sum_k e^{ikx}d(k)$ the distribution from $\mathbf{D}$. Then $\{f(l)\}$ are the Fourier coefficients of the distribution PT_p from $\mathbf{D}$. Equation (51) implies that $P(x)\, T_p(x) \equiv 0$. But the cosine polynomial T_p is strictly positive. Hence we have $P(x) \equiv 0$, which, in turn, leads us to Eq. (52). ∎

COROLLARY 3.1. *Any discrete spline $S_p \in \mathbf{V}_p$ could be uniquely represented through the series*

$$S_p(j)=\sum_{-\infty}^{\infty} S_p(kn)\, L_p(j-kn)$$

which converges locally uniformly.

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